Exercise Sheet Solutions #9

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Felipe Hernández

P1. (Area interpretation of the integral) Let $(X, \mathcal{B}, \mu)$ a s-finite probability space and $f : X \to \mathbb{R}$ a measurable function. Let $R_f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq f(x)\}$. Prove that

(a) If $f : X \to [0, \infty]$ is measurable then $R_f \in \mathcal{B} \otimes \text{Borel}(\mathbb{R})$ and

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_f).$$

where λ is the Lebesgue measure.

> ***Solution:*** We start proving that R_f is $\mathcal{B} \otimes \text{Borel}(\mathbb{R})$ measurable. Indeed, notice that R_f is the the intersection of $X \times [0, \infty)$ (which is measurable for the product sigma algebra) with the inverse image of $[0, \infty)$ though the map taking $(x, t) \in X \times \mathbb{R}$ to $f(x) - t \in \mathbb{R}$. This last map is measurable by being the composition of the map $(y, t) \in \mathbb{R}^2 \to y - t$ with the map $(x, t) \in X \times \mathbb{R} \to (f(x), t)$, where both maps are measurable by definition of the product sigma algebra and by hypothesis of f being measurable.
> As $\mathbb{1}_{R_f}$ is measurable and the spaces $(X, \mathcal{B}, \mu)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ are s-finite, by Fubinni's theorem we have that
>
> $$\begin{aligned}(\mu \overset{\text{c-s}}{\otimes} \lambda)(R_f) &= \int_X \int_{\mathbb{R}} \mathbb{1}_{[0,f(x)]}(t) d\lambda(t) d\mu(x) \\ &= \int_X f(x) d\mu(x).\end{aligned}$$

(b) If $f : X \to \mathbb{R}$ integrable then

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^+}) - (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^-}).$$

> ***Solution:*** We have that $f = f_+ - f_-$ with f_+ and f_- integrable positive functions and
>
> $$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu.$$
>
> By the previous part, this implies that
>
> $$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^+}) - (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^-}),$$
>
> concluding.

P2. Denote λ the Lebesgue measure on $\mathbb{R}$. Show that if $A, B \subseteq$ are Lebesgue measurable sets such that $\lambda(A), \lambda(B) > 0$ then there is $t \in \mathbb{R}$ satisfying $\lambda(A \cap (B - t)) > 0$.

> ***Solution:*** As λ is s-finite and the function $(x, t) \in \mathbb{R}^2 \to \mathbb{1}_A(x)\mathbb{1}_B(x + t) \in \mathbb{R}$ is measurable for the product sigma algebra (as it can be get for measurable operations from the functions $(x, t) \to \mathbb{1}_A(x)$, $(x, t) \to \mathbb{1}_B(t)$, and $(x, t) \to (x, x+t)$ which are measurable), Fubinni's theorem

yields

$$\begin{aligned}\int \lambda(A\cap(B-t))d\lambda(t) &= \int\int \mathbb{1}_A(x)\mathbb{1}_B(x+t)d\lambda(x)d\lambda(t)\\ &= \int\int \mathbb{1}_A(x)\mathbb{1}_B(x+t)d\lambda(t)d\lambda(x)\\ &= \int\int \mathbb{1}_A(x)\mathbb{1}_B(t)d\lambda(t)d\lambda(x) = \lambda(B)\lambda(A) > 0.\end{aligned}$$

Thus, there must be $t\in\mathbb{R}$ such that $\lambda(A\cap(B-t))>0$.

P3. Consider $X=[0,1]$ equipped with the standard topology and $Y=[0,1]$ equipped with the discrete topology. Define $\varphi: C_c(X\times Y)\to\mathbb{C}$ by $\varphi(f)=\sum_y\int_0^1 f(x,y)dx$ where the integral corresponds to the Riemman integral. Let μ be the Radon measure corresponding to φ. Is μ a product of $\lambda|_{\text{Borel}([0,1])}$ and the counting measure ?
Hint: Consider the rectangle $\{0\}\times[0,1]$.

Solution: The answer is no because the product $\rho=\lambda|_{\text{Borel}([0,1])}\otimes\nu$ where ν is the counting measure on $[0,1]$, cannot be regular: If $U\times[0,1]$ is an open set covering $\{0\}\times[0,1]$ then we have that $\rho(U\times[0,1])=\lambda(U)\times\nu([0,1])=\lambda(U)\cdot\infty=\infty$, where we used that $\lambda(U)>0$. Nevertheless, the measure of $\{0\}\times[0,1]$ by definition of product measure is $\rho(\{0\}\times[0,1])=\lambda(\{0\})\times\nu([0,1])=0\cdot\infty=0$.